0.1 Division Algorithm

Notes: For simplicity I am assuming that both polynomials f and g are in the standard form – terms are ordered from highest to lowest degree. This actually is important. It will not work correctly if terms of the polynomials are not ordered by degree.

Note: All terminalogy is standard. However, $Lt(f)$ refers to the leading term of the polynomial f. For example, if $f = -3x^4 + 5x + 23$, then $Lt(f) = -3x^4$.

Given a polynomial $f(x)and$g(x)$,thedivisionalgorithmrepresents$fasqg+r$with$r$\neq f$ and q a polynomial in x. Set the initial value of the quotient and remainder, $q_1 = r = 0$. Begin with $i = 1$, $1 \leq i \leq s$.

1. If the $LT(f) = a_i LT(g)$ for some a_i, a polynomial in x, then set $q = q + a_i$ and $f = f - a_i g_i$. Increment i and reiterate with the new value of f.
2. If the leading term of f is not divisible by any of the leading terms of polynomial in g, then $r = f$ and set $f = f - LT(f)$. Increment i and reiterate with the new value of f.
3. The algorithm ends when there are no terms remaining in f.

Each time this algorithm is iterated, $LT(f)$ is removed resulting in a polynomial of lower degree. Because the degree is decreasing and must remain above 0, the division algorithm must terminate. If the algorithm takes n steps then it produces a quotient polynomial $q(x) = a_1 + ...a_n$.

Example 1

Consider dividing $F = 3x^3 + 5x^2 + x - 1$ by $g = x^2 + 1$.

Initial Conditions: i=1, f=$3x^3 + 5x^2 + x - 1$, $g = x^2 + 1$.

$Lt(g) = x^2$
$Lt(f) = 3x^3 = 3xLt(g)$
$a_i = 3x$
$q = 3x$
$f = f - (3x)g = 3x^3 + 5x^2 + x - 1 - 3x(x^2 + 1) = 3x^3 + 5x^2 + x - 1 - 3x^3 - 3x = 5x^2 - 2x - 1$
increment i

$i = 2$
$Lt(g) = x^2$
$f = 5x^2 - 2x - 1$
$Lt(f) = 5x^2 = 5Lt(g)$
$a_i = 5$
$q = 3x + 5$
$f = f - 5g = 5x^2 - 2x - 1 - 5(x^2 + 1) = 5x^2 - 2x - 1 - 5x^2 - 5 = -2x - 6$
increment i

$i = 3$
$Lt(g) = x^2$

$f = -2x - 1$
$Lt(f) = x$
x^2 does not divide into x. Therefore, $r = -2x - 6$ and the algorithm terminates with the values

$q = 3x + 5$ and $r = -2x - 6$

Therefore, $F = 3x^3 + 5x^2 + x - 1 = (3x + 5)(x^2 + 1) + (-2x - 6)$.

Example 2

Consider dividing $F = 2x^5 - 7x^2 + 1$ by $g = x^2 + 1$.

Initial Conditions: i=1, f=$2x^5 - 7x^2 + 1$, $g = x^2 + 1$.

i=1
$Lt(g) = x^2$
$Lt(f) = 2x^5 = 2x^3 Lt(g)$
$a_i = 2x^3$
$q = 2x^3$
$f = f - (2x^3)g = 2x^5 - 7x^2 + 1 - (2x^3)(x^2 + 1) = 2x^5 - 7x^2 + 1 - 2x^5 - 2x^3 = -2x^3 - 7x^2 + 1$
increment i

i=2
$Lt(g) = x^2$
$f = -2x^3 - 7x^2 + 1$
$Lt(f) = -2x^3 = -2xLt(g)$
$a_i = -2x$
$q = 2x^3 - 2x$
$f = f - -2xLt(g) = -2x^3 - 7x^2 + 1 - -2x(x^2 + 1) = -2x^3 - 7x^2 + 1 + 2x^3 + 2x = -7x^2 + 2x + 1$
increment i

i=3
$Lt(g) = x^2$
$f = -7x^2 + 2x + 1$
$Lt(f) = -7x^2 = -7Lt(g)$
$a_i = -2x$
$q = 2x^3 - 2x - 7$
$f = f - -7Lt(g) = -5x^2 + 2x + 1 - -7(x^2 + 1) = -7x^2 + 2x + 1 + 5x^2 + 5 = 2x + 8$
increment i

i=4
$Lt(g) = x^2$
$f = 2x + 8$
$Lt(f) = 2x$
x^2 does not divide into $2x$. Therefore, $r = 2x + 8$ and the algorithm terminates with the values

$q = 2x^3 - 2x - 7$ and $r = 2x + 8$

Therefore, $F = 2x^5 - 7x^2 + 1 = (2x^3 - 2x - 7)(x^2 + 1) + (2x + 8)$.

The few times I have taught Pre-calculus, the focus is on using this to factor polynomials. Therefore, if g is a factor of the polynomial f, the remainder is 0. This is not the case in either of the examples above. Most importantly, if $g = x - a$ for some constant a, then a is a root of f if and only if the remainder of f divided by g is 0.